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Substitusi yang Merasionalkan

1. Integran memuat $\sqrt[n]{(ax+b)^m}$. Strategi: gunakan subsitusi $u = \sqrt[n]{ax+b}$ atau $u^n = ax+b$. Diperoleh

$$\sqrt[n]{(ax+b)^m} = u^m, u^n = ax+b \text{ dan } adx = nu^{n-1}$$

Contoh-contoh

Contoh $\int x\sqrt{x+e}dx$. Misalkan $u = \sqrt{x+e}$ atau $x = u^2 - e$. Maka $2udu = dx$. Jadi,

$$\int x\sqrt{x+e}dx = \int \left(u^2 - e\right) u\,(2udu) = \int \left(2u^4 - 2u^2 e\right) du = \frac{2u^5}{5} - \frac{2eu^3}{3} + C$$
$$= \frac{2\,(x+e)^{\frac{5}{2}}}{5} - \frac{2e\,(x+e)^{\frac{3}{2}}}{3} + C$$

Contoh $\int \frac{xdx}{\sqrt[3]{(x+4)^2}}$. Misalkan $u = \sqrt[3]{x+4} = (x+4)^{\frac{1}{3}}$. Maka $\sqrt[3]{(x+4)^2} = u^2, u^3 = x+4$ (atau $x = u^3 - 4$), dan $dx = 3u^2du$.

$$\int \frac{xdx}{\sqrt[3]{(x+4)^2}} = \int \frac{\left(u^3-4\right)3u^2du}{u^2} = 3\int \left(u^3 - 4\right) du = \frac{3}{4}u^4 - 12u + C$$
$$= \frac{3}{4}(x+4)^{\frac{4}{3}} - 12\,(x+4)^{\frac{1}{3}} + C$$

2. Integran memuat $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$, dengan $a > 0$. Substitusi yang digunakan

   (a) Untuk $\sqrt{a^2-x^2}$, $x = a\sin t, -\frac{\pi}{2} \le t \le \frac{\pi}{2}, dx = a\cos t\, dt$. Hasil substitusi: $\sqrt{a^2-x^2} = \sqrt{a^2\cos^2 t} = a\cos t$, karena $\cos t \ge 0$.

   (b) Untuk $\sqrt{a^2+x^2}$, $x = a\tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, dx = a\sec^2 t\ dt$. Hasil substitusi: $\sqrt{a^2-x^2} = \sqrt{a^2\sec^2 t} = a\sec t$, karena $a\sec t \ge 0$.

   (c) Untuk $\sqrt{x^2-a^2}$, $x = a\sec t, dx = a\sec t\tan t\ dt, 0 \le t \le \pi, t \ne \frac{\pi}{2}$ yaitu

$$\begin{bmatrix} 0 < t < \frac{\pi}{2} & \text{jika } t > a \\ \frac{\pi}{2} < t < \pi & \text{jika } t < -a \end{bmatrix}$$

   Hasil substitusi:

$$\sqrt{a^2-x^2} = |a\tan t| = a\,|\tan t| = \pm a\tan t = \begin{cases} a\tan t, & \text{jika } 0 \le t < \frac{\pi}{2} \\ -a\tan t, & \text{jika } -\frac{\pi}{2} < t \le \pi \end{cases}$$

   Alternatif: $\sqrt{x^2-a^2}$, $x = a\sec t, dx = a\sec t\tan t\ dt, 0 \le t < \frac{\pi}{2}$ atau $\pi \le t < \frac{3\pi}{2}$. Maka $\tan t \ge 0$. Dengan demikian, $\sqrt{x^2-a^2} = a\tan t$.

   Beberapa contoh.

   Contoh $\int \frac{dx}{\sqrt{x^2+6}}$. Misalkan $x = \sqrt{6}\tan t$. Maka $dx = \sqrt{6}\sec^2 tdt, -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\int \frac{dx}{\sqrt{x^2+6}} = \int \frac{\left(\sqrt{6}\sec^2 t\right)dt}{\sqrt{6}\sec t} = \int \sec tdt = \ln|\sec t + \tan t| + C$$
$$= \ln\left|\frac{\sqrt{6}}{\sqrt{x^2+6}} + \frac{x}{\sqrt{6}}\right| + C.$$

Contoh $\int \frac{dx}{x^3\sqrt{x^2-9}}$. Misalkan $x = 3\sec t$. Jika $x > 3$

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-9}} &= \int \frac{3\sec t\tan t dt}{27\sec^3 t.3\tan t} = \frac{1}{27}\int \frac{dt}{\sec^2 t} = \frac{1}{27}\int \cos^2 t dt \\
&= \frac{1}{27}\int \frac{1+\cos 2t}{2} dt = \frac{1}{54}\left(t + \frac{\sin 2t}{2}\right) + C \\
&= \frac{1}{54}\left(t + \sin t\cos t\right) + C = \frac{1}{54}\left(\sec^{-1}\frac{x}{3} + \frac{\sqrt{x^2-9}}{x}\frac{3}{x}\right) + C
\end{aligned}$$

Jika $x < -3$,

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-9}} &= \int \frac{3\sec t\tan t dt}{27\sec^3 t.3\,(-1)\tan t} = -\frac{1}{27}\int \frac{dt}{\sec^2 t} \\
&= -\frac{1}{54}\left(\sec^{-1}\frac{x}{3} + \frac{\sqrt{x^2-9}}{x}\frac{3}{x}\right) + C
\end{aligned}$$

3. Integran memuat $x^2 + px + q$ dibawah tanda akar. Lakukan melengkapkan kuadrat untuk memperoleh bentuk

$$\begin{aligned}
x^2 + px + q &= \beta^2 - (x+\alpha)^2 \text{ atau} \\
x^2 + px + q &= (x+\alpha)^2 - \beta^2 \text{ atau} \\
x^2 + px + q &= (x+\alpha)^2 + \beta^2
\end{aligned}$$

integral menjadi salah satu kasus 2.

Contoh $\int \frac{xdx}{\sqrt{6x-x^2}}$. Melengkapkan kuadrat: $6x - x^2 = -\left(x^2 - 6x\right) = -\left((x-3)^2 - 9\right) = 9 - (x-3)^2$. Misalkan $x - 3 = 3\sin t, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Maka $9x - x^2 = 9 - 9\sin^2 t = 9\cos^2 t$. Jadi,

$$x = 3 + 3\sin t, \quad t = \sin^{-1}\left(\frac{x-3}{3}\right), \quad \sqrt{9x - x^2} = 3\cos t, \quad dx = 3\cos t\ dt.$$

Jadi,

$$\begin{aligned}
3\sin^{-1}\left(\frac{x-3}{3}\right) - \sqrt{6x-x^2} = \int \frac{xdx}{\sqrt{6x-x^2}} &= \int \frac{(3+3\sin t)\,3\cos t dt}{3\cos t} = \int (3+3\sin t)\,dt \\
&= 3t - 3\cos t + C = 3\sin^{-1}\left(\frac{x-3}{3}\right) - \sqrt{6x-x^2} + C
\end{aligned}$$

Contoh $\int \frac{3xdx}{\sqrt{x^2+4x-5}}$. $x^2 + 4x - 5 = (x+2)^2 - 4 - 5 = (x+2)^2 - 9$. Misalkan $x + 2 = 3\sec t$. Dengan demikian

$$x = 3\sec t - 2, \sqrt{x^2+4x-5} = \pm 3\tan t, dx = 3\sec t\tan t\ dt.$$

Selanjutnya, jika $x + 2 > 3$, maka $\sqrt{x^2+4x-5} = 3\tan t$.

$$\begin{aligned}
\int \frac{3xdx}{\sqrt{x^2+4x-5}} &= \int \frac{3\,(3\sec t - 2)\,3\sec t\tan t\ dt}{3\tan t} = 3\int (3\sec t - 2)\sec t dt \\
&= 9\int \sec^2 t\ dt - 6\int \sec t\ dt = 9\tan t - 6\ln|\sec t + \tan t| + C \\
&= 3\sqrt{x^2+4x-5} - 6\ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C
\end{aligned}$$

Dengan cara serupa, jika $x + 2 < -3$, maka $\sqrt{x^2+4x-5} = -3\tan t$, dan oleh karena itu

$$\int \frac{3xdx}{\sqrt{x^2+4x-5}} = -3\sqrt{x^2+4x-5} + 6\ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C$$