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Tehnik Pengintegralan: Integral Trigonometri

Bentuk-bentuk integral yang akan dibahas:

1. $\int \sin^n x \cos^m x dx$.
2. $\int \sin nx \sin mx dx$, $\int \cos nx \cos mx dx$, $\int \sin nx \cos mx dx$.
3. $\int \tan^n x \sec^m x dx$, $\int \cot^n x \csc^m x dx$.

Strategi pengintegraan untuk tiap bentuk:

1. $\int \sin^n x \cos^m x dx$.

Kasus n ganjil atau m ganjil. Dalam hal ini, identitas yang digunakan adalah

$$\sin^2 x + \cos^2 x = 1$$

Jika n ganjil, pisahkan satu faktor $\sin x$ dan gunakan substitusi $u = \cos x$. Gunakan identitas $\sin^2 x = 1 - \cos^2 x$ untuk mengganti semua $\sin x$ dengan $\cos x$. Contoh:

$$\begin{aligned}\int \sin^5 x \cos^2 x dx &= \int \sin^4 x \cos^2 x \sin dx = \int \sin^4 x \cos^2 x d(-\cos x) = -\int \sin^4 x \cos^2 x d \cos x \\ &= -\int \left(\sin^2 x\right)^2 \cos^2 x d \cos x = -\int \left(1-\cos^2 x\right)^2 \cos^2 x d \cos x = -\int \left(1-u^2\right)^2 u^2 du \\ &= -\frac{1}{7}u^7 + \frac{2}{5}u^5 - \frac{1}{3}u^3 + C = -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C\end{aligned}$$

Strategi unutk m ganjil adalah serupa: pisahkan satu faktor $\cos x$ dan gunakan substitusi $u = \sin x$. Gunakan identitas $\cos^2 x = 1 - \sin^2 x$ untuk mengganti semua $\cos x$ dengan $\sin x$. Contoh:

$$\begin{aligned}\int \sin^3 x \cos^3 x dx &= \int \sin^3 x \cos^2 x \cos dx = \int \sin^3 x \cos^2 x d \sin x = \int \sin^3 x \left(1 - \sin^2 x\right) d \sin x \\ &= \int u^3 \left(1-u^2\right) du = \int \left(u^3 - u^5\right) du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C \\ &= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.\end{aligned}$$

Kasus m dan n genap. Dalam hal ini, identitas yang digunakan adalah

$$\cos^2 x = \frac{1+\cos 2x}{2} \text{ dan } \sin^2 x = \frac{1-\cos 2x}{2}$$

Kedua identitas digunakan untuk 'menurunkan pangkat dari cos dan sin sehingga kompleksitas juga turun. Contoh:

$$\begin{aligned}
\int \sin^2 x \cos^2 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)\, dx \\
&= \frac{1}{8}\int \left(1-\cos^2 2x\right) dx = \frac{1}{8}\int \left(1-\left(\frac{1+\cos 4x}{2}\right)\right) dx = \frac{1}{16}\int (1-\cos 4x)\, dx \\
&= \frac{x}{16} - \frac{1}{64}\sin 4x + C \\
\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2\, dx \\
&= \frac{1}{8}\int \left((1-\cos 2x)\left(1+2\cos 2x+\cos^2 2x\right)\right) dx \\
&= \frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right) dx \\
&= \frac{1}{8}\left(\int dx + \int \cos 2x dx - \int \cos^2 2x dx - \int \cos^3 2x dx\right) \\
&= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \frac{1}{2}\int (1+\cos 4x)\, dx - \frac{1}{2}\int \cos^2 2x d\sin 2x\right) \\
&= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \frac{1}{2}\int (1+\cos 4x)\, dx - \frac{1}{2}\int \left(1-\sin^2 2x\right) d\sin 2x\right) \\
&= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right) - \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)\right) + C
\end{aligned}$$

Maka,

$$\begin{aligned}
\int \sin^2 x \cos^4 x dx &= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right) - \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)\right) + C \\
&= \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C
\end{aligned}$$

2. $\int \sin nx \sin mx dx, \int \cos nx \cos mx dx, \int \sin nx \cos mx dx$. Gunakan identitas-identitas

$$\begin{aligned}
\sin nx \sin mx &= \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x \\
\cos nx \cos mx &= \frac{1}{2}\cos(n-m)x + \frac{1}{2}\cos(n+m)x \\
\sin nx \cos mx &= \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x
\end{aligned}$$

Sebagai contoh

$$\begin{aligned}
\int \sin\sqrt{3}x \sin\frac{x}{2} dx &= \frac{1}{2}\int \left(\cos\left(\sqrt{3}-\frac{1}{2}\right)x - \cos\left(\sqrt{3}+\frac{1}{2}\right)x\right) dx \\
&= \frac{1}{2}\int \cos\left(\sqrt{3}-\frac{1}{2}\right) x dx - \frac{1}{2}\int \cos\left(\sqrt{3}+\frac{1}{2}\right) x dx \\
&= \frac{1}{2}\frac{1}{\sqrt{3}-\frac{1}{2}}\sin\left(\sqrt{3}-\frac{1}{2}\right)x - \frac{1}{2}\frac{1}{\sqrt{3}+\frac{1}{2}}\sin\left(\sqrt{3}+\frac{1}{2}\right)x + C.
\end{aligned}$$

3. $\int \tan^m x \sec^n x dx, \int \cot^m x \csc^n x dx$. Beberapa fakta yang digunakan:

$$\tan^2 x = \sec^2 x - 1 \quad \text{dan} \qquad \cot^2 x = \csc^2 x - 1$$

$$\int \sec^2 x dx = \tan x + C \ \left(d\tan x = \sec^2 x dx\right) \ \text{dan} \ \int \csc^2 x dx = -\cot x + C \ \left(-d\cot x = \csc^2 x dx\right)$$

$$\int \tan x dx = -\ln|\cos x| + C \text{ dan } \int \cot x dx = \ln|\sin x| + C$$

$$\int \sec x \tan x dx = \sec x + C \text{ dan } \int \csc x \cot x dx = -\csc x + C$$

(a) **Kasus** $n = 0$.

i. **Subkasus** m genap. Pisahkan $\tan^2 x$ atau $\cot^2 x$, kemudian gunakan identitas $\tan^2 x = \sec^2 x - 1$ atau $\cot^2 x = \csc^2 x - 1$. Selesaikan integral dengan substitusi peubah $u = \tan x$ (untuk $\int \tan^n x dx$) atau $u = \cot x$ (untuk $\int \cot^n x dx$) dan integral $\int \sec^2 x dx = \tan x + C$ dan $\int \csc^2 x dx = -\cot x + C$

$$\begin{aligned}
\int \tan^4 x dx &= \int \tan^2 x \left(\sec^2 x - 1\right) = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\
&= \int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right) dx = \int \tan^2 x d \tan x - \int \tan x dx + \int dx \\
\int \tan^4 x dx &= \frac{1}{3} \tan^3 x - \tan x + x + C.
\end{aligned}$$

ii. **Subkasus** m ganjil. Pisahkan $\tan^2 x$ atau $\cot^2 x$, kemudian gunakan identitas $\tan^2 x = \sec^2 x - 1$ atau $\cot^2 x = \csc^2 x - 1$. Selesaikan integral dengan substitusi peubah $u = \tan x$ dan $du = \csc^2 x$ (untuk $\int \tan^n x dx$) atau $u = \cot x$ dan $du = -\csc^2 x dx$(untuk $\int \cot^n x dx$) serta (dalam hal n ganjil) integral $\int \tan x dx = -\ln|\cos x| + C$ dan $\int \cot x dx = \ln|\sin x| + C$. Strategi

$$\begin{aligned}
\int \cot^5 x dx &= \int \cot^3 x \left(\csc^2 x - 1\right) dx = \int \cot^3 x \csc^2 x dx - \int \cot^3 x dx = \int \cot^3 x \csc^2 x dx - \int \cot x \left(\csc^2 x\right. \\
&= \int \cot^3 x \csc^2 x dx - \int \cot x \csc^2 x dx + \int \cot x dx \\
&= \int \cot^3 x (-1) \, d(\cot x) - \int \cot x (-1) \, d \cot x + \ln|\sin x| + C \\
&= -\frac{1}{4} \cot^4 x + \frac{1}{2} \cot^2 x + \ln|\sin x| + C.
\end{aligned}$$

(b) $\int \tan^m x \sec^n x dx, \int \cot^m x \csc^n x dx$.

i. **Subkasus** m ganjil: Pisahkan $\sec x \tan x$. Gunakan hubungan $\tan^2 x = \sec^2 x - 1$ untuk menulis $\tan^{n-1} x$ (dalam $\sec x$ dan $d \sec x = \sec x \tan x dx$, untuk menyelesaikan integral dengan substitusi $u = \sec x$. Lakukan strategi serupa untuk $\int \cot^m x \csc^n x dx$.

$$\begin{aligned}
\int \tan^5 x \sec^3 x dx &= \int \tan^4 x \sec^2 x \sec x \tan x dx = \int \left(\tan^2 x\right)^2 \sec^2 x d \sec x = \int \left(\sec^2 x - 1\right)^2 \sec^2 x d \sec x \\
&= \int \left(u^2 - 1\right)^2 u^2 du = \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C \\
&= \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C.
\end{aligned}$$

ii. **Subkasus** n genap. Pisahkan $\sec^2 x$ dan bentuk $\sec^2 x dx = d \tan x$. Gunakan $\sec^2 x = \tan^2 x + 1$ untuk menuliskan $\sec^{m-2} x$ dalam $\tan x$. Selesaikan integral dengan substitusi $u = \tan x$. Lakukan strategi serupa untuk $\int \cot^m x \csc^n x dx$.

$$\begin{aligned}
\int \tan^2 x \sec^4 x dx &= \int \tan^2 x \sec^2 x \sec^2 x dx = \int \tan x \left(\tan^2 x + 1\right) d \tan x \\
&= \int u \left(u^2 + 1\right) du = \frac{u^4}{4} + \frac{u^2}{2} + C = \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + C
\end{aligned}$$

$$\int \cot^4 x \csc^2 x dx = \int \cot^4 x (-1) \, d \cot x = -\int u^4 du = -\frac{1}{5} \cot^5 x + C$$

Peta penyelesaian kasus sejauh ini:

$m\|n$	n genap	n ganjl
m genap	✓	×
m ganjil	✓	✓

iii. **Subkasus** m genap dan n ganjil. (Pengayaan) **Tidak ada** strategi umum. Beberapa integral

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x dx = -\ln|\csc x + \cot x| + C$$

Penjelasan:

$$\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\left(\sec^2 x + \sec x \tan x\right) dx}{\sec x + \tan x}$$

Misalkan $u = \sec x + \tan x$. Maka $du = \left(\sec x \tan x + \sec^2 x\right) dx$. Maka $\int \sec x dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$. Dengan cara serupa,[erlihatkan bahwa $\int \csc x dx = -\ln|\csc x + \cot x| + C$.

Contoh lain

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx = \int \sec x d \tan x = \sec x \tan x - \int \tan x d \sec x \\
&= \sec x \tan x - \int \tan x \sec x \tan x dx = \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|
\end{aligned}$$

Maka

$$\int \sec^3 x dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Satu contoh lagi.

$$\begin{aligned}
\int \tan^2 x \sec x dx &= \int \tan x \sec x \tan x = \int \tan x d \sec x \\
&= \int \left(\sec^2 x - 1\right) \sec x dx = \int \sec^3 x dx - \int \sec x dx \\
&= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C
\end{aligned}$$